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EFFECT OF TRANSVERSE SHEAR DEFORMATION AND ROTARY INERTIA ON THE NATURAL FREQUENCIES OF RECTANGULAR PLATES WITH CUTOUTS

H.-P. LEE, S. P. LIM and S. T. CHOW
Department of Mechanical and Production Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511

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Abstract—A numerical method based on the Rayleigh principle is applied in conjunction with the simplified Reissner–Mindlin theory which involves only a single variable, the transverse displacement, for predicting the fundamental frequencies of rectangular plates with rectangular cutouts. The predicted results are compared with the finite elements results and the reported works available in the literature. It is concluded that the effect of rotary inertia can only be neglected for orthotropic plates with high degrees of orthotropy. For isotropic plates, the effect of rotary inertia becomes more pronounced with respect to the effect of transverse shear deformation with increased cutout size.

NOTATION

a, b	dimension of the plate
c, d	dimension of the cutout
r_1, r_2	x coordinates of the corners of the cutout
s_1, s_2	y coordinates of the corners of the cutout
h	thickness of the plate
k	shear correction factor $= \frac{5}{6}$
x, y	rectangular coordinate axes
D	flexural rigidity of an isotropic plate
D_x, D_y	flexural rigidities of an orthotropic plate
D_1	$= \nu_{xy} D_y = \nu_{yx} D_x$
D_{xy}	torsional rigidity of an orthotropic plate
E	Young's modulus of an isotropic plate
E_x, E_y	Young's moduli of an orthotropic plate
G	shear modulus of an isotropic plate
G_{xy}, G_{yz}, G_{xz}	shear moduli of an orthotropic plate
Q_x, Q_y	shear forces per unit length
S_x, S_y	shear stiffness of an orthotropic plate
W, W_n	transverse deflection
ρ	mass density
ω	angular frequency
λ	non-dimensional frequency parameter
V	potential energy
T	kinetic energy
ν	Poisson's ratio of isotropic plates
ν_{xy}	Poisson's ratio of orthotropic plates
ϕ_x, ϕ_y	rotations.

1. INTRODUCTION

The past research in the effects of transverse shear deformation and rotary inertia on the natural frequencies of plates have been mostly confined to plates without any cutout. Numerical results for simply-supported square plates with central rectangular cutouts were reported by Reddy (1982) and Tham *et al.* (1986) based on the finite element method. Aksu (1984) presented an analysis based on the finite difference analysis. All these approaches used Reissner–Mindlin plate theory to investigate the linear vibration of rectangular plates. No known analytical results exist to the limited knowledge of the authors.

The aforementioned numerical methods need substantial computational effort. Frequently, estimates of the natural frequencies of the fundamental and the first few higher modes are sufficient for engineering applications. This paper attempts to present a simple numerical method based on the Rayleigh principle and the simplified Reissner–Mindlin theory, originally proposed by Speare and Kemp (1977) for solving the plate bending problem, for predicting the linear natural frequencies of rectangular plates with rectangular cutouts. The method is illustrated for simply supported rectangular plates with central cutouts. The predicted results are compared with the reported finite element results and results generated by a finite element software package PAFEC.

2. ANALYSIS

The plate under consideration is a thin simply-supported rectangular plate of size $a \times b$ having a central rectangular cutout of size $c \times d$ as shown in Fig. 1. The plates are divided into smaller sub-domains based on the mode shapes and the locations of the cutouts. The sub-domain technique was first proposed by the authors (1986, 1987, 1990) for predicting the natural frequencies of rectangular plates with cutouts and cracks based on the Classical plate theory and the Rayleigh method. In order to apply the Rayleigh method for predicting the natural frequencies, the kinetic energy and potential energy of the plate have to be formulated in terms of a single variable, W, the transverse deflection of the plate. The application of the Rayleigh–Ritz technique to the original Reissner–Mindlin formulation for rectangular plates without cutout was reported by Dawe and Roufaeil (1980) and Roufaeil and Dawe (1982). Although formulation based on the Reissner–Mindling theory will generate more accurate results, the formulation and subsequent computation are more involved since the theory employs three independent variables namely the transverse deflection, W, and rotations ϕ_x and ϕ_y.

The present formulation based on the simplified Reissner–Mindlin theory, proposed by Speare and Kemp (1977), only involves a single variable W. For an orthotropic plate with the principal axes of orthotropy coincide with the x and y directions of the plate, the bending moments M_x, M_y, and the twisting moment M_{xy} are as follows

$$M_x = D_x \frac{\partial \phi_x}{\partial x} + D_1 \frac{\partial \phi_y}{\partial y}, \tag{1}$$

$$M_y = D_y \frac{\partial \phi_y}{\partial y} + D_1 \frac{\partial \phi_x}{\partial x}, \tag{2}$$

$$M_{xy} = -D_{xy}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \tag{3}$$

in which

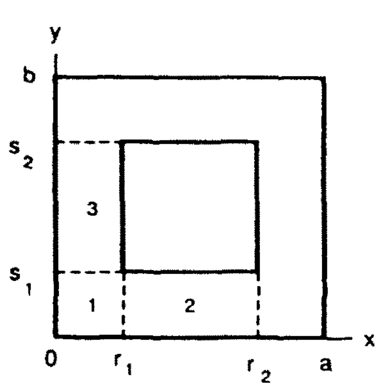

Fig. 1. A rectangular plate with a central rectangular cutout.

$$\phi_x = -\frac{\partial W}{\partial x} + \frac{Q_x}{S_x}, \quad S_x = kG_{xz}h, \tag{4}$$

$$\phi_y = -\frac{\partial W}{\partial y} + \frac{Q_y}{S_y}, \quad S_y = kG_{yz}h, \tag{5}$$

$$Q_x = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} + \frac{\rho h^3}{12}\frac{\partial^2 \phi_x}{\partial t^2}, \tag{6}$$

$$Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} + \frac{\rho h^3}{12}\frac{\partial^2 \phi_y}{\partial t^2}. \tag{7}$$

The shear correction factor k is taken to be $\frac{5}{6}$ as proposed by Speare and Kemp (1977) in their formulation of simplified Reissner theory for plate bending. If the rotary inertia terms are neglected, Q_x and Q_y can be approximated by the following expressions:

$$\begin{aligned} Q_x = & -D_x\frac{\partial^3 W}{\partial x^3} - (D_1 + 2D_{xy})\frac{\partial^3 W}{\partial x\,\partial y^2} - \frac{D_x^2}{S_x}\frac{\partial^5 W}{\partial x^5} \\ & - \left(\frac{D_x(D_1+3D_{xy})}{S_x} + \frac{(D_1+D_{xy})(D_1+2D_{xy})}{S_y}\right)\frac{\partial^5 W}{\partial x^3\,\partial y^2} \\ & - (D_1+2D_{xy})\left(\frac{D_{xy}}{S_x} + \frac{D_1+D_{xy}}{S_y}\right)\frac{\partial^5 W}{\partial x\,\partial y^4} \end{aligned} \tag{8}$$

$$\begin{aligned} Q_y = & -D_y\frac{\partial^3 W}{\partial y^3} - (D_1 + 2D_{xy})\frac{\partial^3 W}{\partial x^2\,\partial y} - \frac{D_y^2}{S_y}\frac{\partial^5 W}{\partial y^5} \\ & - \left(\frac{D_y(D_1+3D_{xy})}{S_y} + \frac{(D_1+D_{xy})(D_1+2D_{xy})}{S_x}\right)\frac{\partial^5 W}{\partial x^2\,\partial y^3} \\ & - (D_1+2D_{xy})\left(\frac{D_{xy}}{S_y} + \frac{D_1+D_{xy}}{S_x}\right)\frac{\partial^5 W}{\partial x^4\,\partial y}. \end{aligned} \tag{9}$$

Besides rotary inertial terms, in h^4 and higher powers of h have also been neglected.

The potential energy, V and the kinetic energy, T of the plate are evaluated by expressing ϕ_x and ϕ_y in terms of W and substituting into the following expressions (Magrab, 1977):

$$\begin{aligned} V = \frac{1}{2}\iint\Bigg[& D_x\left(\frac{\partial \phi_x}{\partial x}\right)^2 + D_y\left(\frac{\partial \phi_y}{\partial y}\right)^2 + 2D_1\frac{\partial \phi_x}{\partial x}\frac{\partial \phi_y}{\partial y} + D_{xy}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)^2 \\ & + S_x\left(\phi_x + \frac{\partial W}{\partial x}\right)^2 + S_y\left(\phi_y + \frac{\partial W}{\partial y}\right)^2\Bigg]\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{10}$$

$$T = \tfrac{1}{2}\rho h\omega^2\iint W^2\,\mathrm{d}x\,\mathrm{d}y. \tag{11}$$

To include the effect of rotary inertia, the expression for the kinetic energy is modified as

$$T = \tfrac{1}{2}\rho h\omega^2 \int\int \left[W^2 + \frac{h^2}{12}\phi_x^2 + \frac{h^2}{12}\phi_y^2 \right] \mathrm{d}x\,\mathrm{d}y. \tag{12}$$

Q_x and Q_y are first evaluated by neglecting the rotary terms. The resulting estimate for the natural frequency of the plate is then substituted into eqns (8) and (9) to modify the values of q_x and Q_y. ϕ_x and ϕ_y are then re-evaluated. This iteration is performed until the resulting natural frequency converges with an adequate order of accuracy. The expressions for the iterations are as follows:

$$[\phi_x]_{n+1} \text{ iteration} = \left[\frac{1}{1 + \dfrac{\rho h^3 \omega^2}{12 S_x}} \right] [\phi_x]_n \text{ iteration}, \tag{13}$$

$$[\phi_y]_{n+1} \text{ iteration} = \left[\frac{1}{1 + \dfrac{\rho h^3 \omega^2}{12 S_y}} \right] [\phi_y]_n \text{ iteration}. \tag{14}$$

For an isotropic plate, eqns (8) and (9) can be simplified to

$$Q_x = -D\left(\frac{\partial^3 W}{\partial x^3} + \frac{\partial^3 W}{\partial x \partial y^2}\right) - \frac{h^2}{5(1-\nu)} D \left(\frac{\partial^5 W}{\partial x^5} + 2\frac{\partial^5 W}{\partial x^3 \partial y^2} + \frac{\partial^5 W}{\partial x \partial y^4}\right), \tag{15}$$

$$Q_y = -D\left(\frac{\partial^3 W}{\partial y^3} + \frac{\partial^3 W}{\partial x^2 \partial y}\right) - \frac{h^2}{5(1-\nu)} D \left(\frac{\partial^5 W}{\partial y^5} + 2\frac{\partial^5 W}{\partial x^2 \partial y^3} + \frac{\partial^5 W}{\partial x^4 \partial y}\right). \tag{16}$$

The potential and kinetic energy for an isotropic plate can thus be expressed with these two expressions for Q_x and Q_y.

The simplified Reissner–Mindlin theory enables the potential and the kinetic energy to be expressed in terms of the single variable, W. The method developed by the authors (1986) for predicting the natural frequencies of rectangular plates with cutouts can thus be applied to rectangular plates taking into account both the transverse shear deformation and rotary inertia. Numerical results based on eqns (10) and (11) take into consideration the effect of transverse shear deformation but neglect the effect of rotary inertia. Both of these effects are taken into account by applying eqns (10) and (12).

Simply-supported rectangular plates

The method is illustrated for predicting the fundamental frequencies of simply-supported rectangular plates with centrally located cutouts. For the sub-domains of the rectangular plate shown in Fig. 1, the assumed deflection functions are as follows:

$$W_1 = A_1 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b},$$

$$W_2 = A_2 \frac{y}{s_1} \sin\frac{\pi x}{a}, \quad A_2 = A_1 \sin\frac{\pi s_1}{b},$$

$$W_3 = A_3 \frac{x}{r_1} \sin\frac{\pi y}{b}, \quad A_3 = A_1 \sin\frac{\pi r_1}{a}. \tag{17}$$

A_2 and A_3 are determined by matching the maximum deflections of adjacent sub-domains at the lower left-hand corner of the cutout. The assumed deflection functions satisfy the following boundary conditions at the simply-supported edges

$$x = 0, a, \quad W = \phi_y = 0, \tag{18}$$

$$y = 0, b, \quad W = \phi_x = 0. \tag{19}$$

The conditions for the in-plane displacements, u and v, are not specified as the in-plane displacements of the mid-plane are neglected for the present linear analysis.

The kinetic energy and potential energy are evaluated for each sub-domain using eqns (10), (11) and (12). The total potential energy and kinetic energy of the plate are taken to be the respective sums of each sub-domain. The natural frequency is then obtained by equating the total potential energy to the total kinetic energy. The frequency is expressed in terms of a non-dimensional frequency parameter given by

$$\lambda = \omega a^2 \sqrt{\frac{\rho h}{D_y}}, \tag{20}$$

D_y is replaced by D for isotropic plates.

Finite element results

The finite element results of rectangular plates allowing for both the transverse shear deformation and rotary inertia are generated by a computer software package PAFEC. A mesh of between 60 and 80 eight-node facet shell elements based on the Reissner–Mindlin plate theory is used in the finite element analysis. In order to compare the results with the reported finite element results of Reddy (1982) which neglect the effect of rotary inertia, the boundary conditions are taken to be that specified by Reddy (1982), namely,

$$x = 0, a, \quad W = \phi_y = v = 0, \tag{21}$$

$$y = 0, b, \quad W = \phi_x = u = 0. \tag{22}$$

3. RESULTS AND DISCUSSION

The predicted results and the corresponding finite element results for the fundamental mode of a simply-supported isotropic ($\nu = 0.3$) square plate with a central square cutout are presented in Table 1. Although the assumed function is only a one-term approximation,

Table 1. Frequency parameter λ for an isotropic square plate with a square cutout

				Present method	
Cutout size $c \times d$	Thickness $\times a$	PAFEC	Reddy	Shear	Shear and rotary inertia
$0.0a \times 0.0a$	0.001	19.751	19.752	19.739	19.739
	0.100	19.119	19.077	19.205	19.049
	0.200	17.494	17.458	17.835	17.261
$0.2a \times 0.2a$	0.001	19.120	19.200	18.901	18.901
	0.100	18.581	18.679	18.470	18.279
	0.200	17.257	17.452	17.350	16.671
$0.4a \times 0.4a$	0.001	20.732	20.807	20.556	20.556
	0.100	20.005	20.246	20.101	19.781
	0.200	18.448	19.163	18.975	17.879
$0.5a \times 0.5a$	0.001	23.235	23.515	23.329	23.329
	0.100	22.390	22.804	22.802	22.308
	0.200	20.318	21.554	21.533	19.881
$0.6a \times 0.6a$	0.001	28.241	28.453	28.491	28.491
	0.100	26.618	27.379	27.844	26.949
	0.200	23.448	25.668	26.301	23.434
$0.8a \times 0.8a$	0.001	57.452	57.512	58.847	58.846
	0.100	46.658	51.465	57.615	51.424
	0.200	32.889	44.069	54.564	39.079

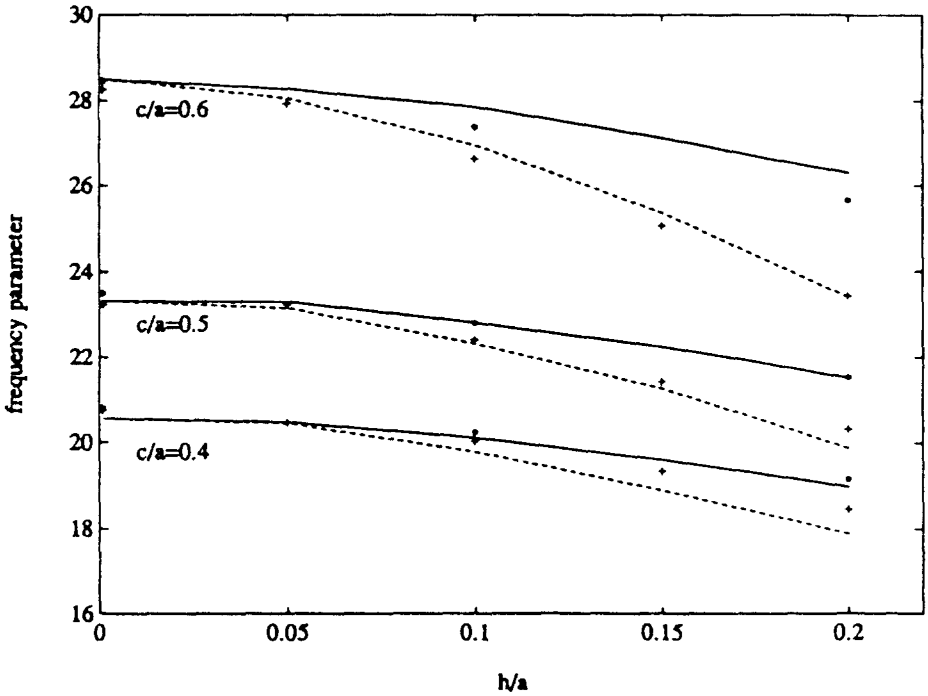

Fig. 2. Fundamental frequency parameters for an isotropic square plate with a square cutout. "——", shear, "- - -", shear and rotary inertia, "•", Reddy, "+", PAFEC.

the predicted results compare favorably with the corresponding finite element results for square plates with $h/a < 0.2$ and for square cutout with $c, d < 0.7a$. For a thick plate with a large square cutout, the thickness of the plate can be of the same order, or even larger than the remaining span of the side strip. The assumption of a plate structure no longer holds for such a "plate".

The predicted results as well as the finite element results show that the effect of rotary inertia becomes more pronounced with respect to the effect of transverse shear deformation for increased size of cutout. The inclusion of rotary inertia reduces the natural frequencies for thick plates ($h/a > 0.1$) with large cutouts ($c, d > 0.5a$) by an additional amount comparable to, or even larger than the reduction caused by the shear deformation alone. The discrepancies between the predicted results with and without the rotary inertia for a square plate increase with increased thickness of the plate. The trends are reflected in Fig. 2.

For an orthotropic plate with the principal axes of orthotropy parallel to the plate edges, the present formulation is only applicable for a range of thicknesses depending on the orthoropic properties and the aspect ratio of the plate. The predicted natural frequencies show an increasing trend for the thickness larger than a certain critical value. It is not surprising as the simplified Reissner–Mindlin theory is only an approximation ignoring terms in h^4 and higher powers of h. The error is expected to increase with increased thickness and increased E_x/E_y ratio. For a simply-supported orthotropic plate with $G_{xy} = G_{xz} = G_{yz} =$

Table 2. The critical thickness for the application of the present formulation, $G/E_y = 0.5$, $\nu_x = 0.25$

E_x/E_y	Critical square plate	Thickness $\times b$ Rectangular plate with $a/b = 2$
3	>0.40	>0.40
10	0.17	0.33
20	0.12	0.24
30	0.10	0.20
40	0.09	0.17

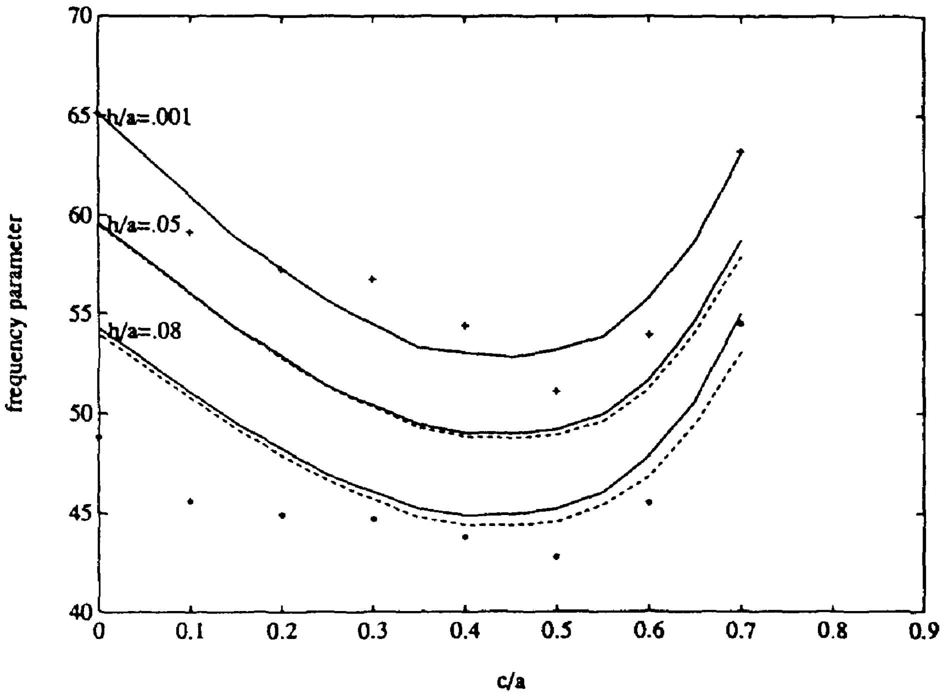

Fig. 3. Fundamental frequency parameters for a graphite-epoxy square plate with a square cutout, "——", shear, "- - -", shear and rotary inertia, "•", $h/a = 0.1$, Reddy, "+", $h/a = 0.01$, Reddy.

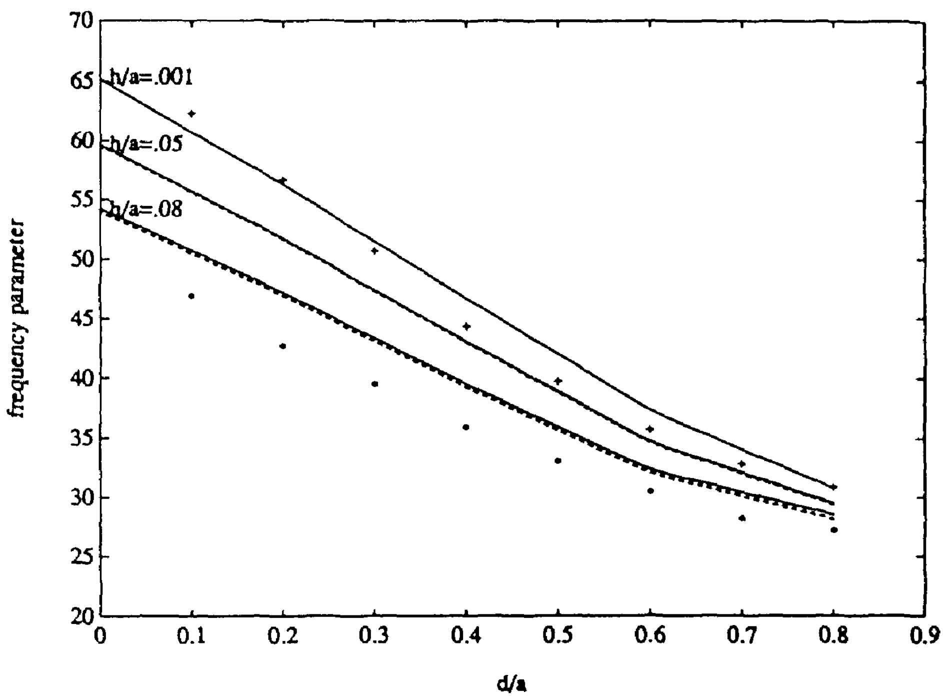

Fig. 4. Fundamental frequency parameters for a graphite-epoxy square plate having a central rectangular cutout with $d/c = 2$, "——", shear, "- - -", shear and rotary inertia, "•", $h/a = 0.1$, Reddy, "+", $h/a = 0.001$, Reddy.

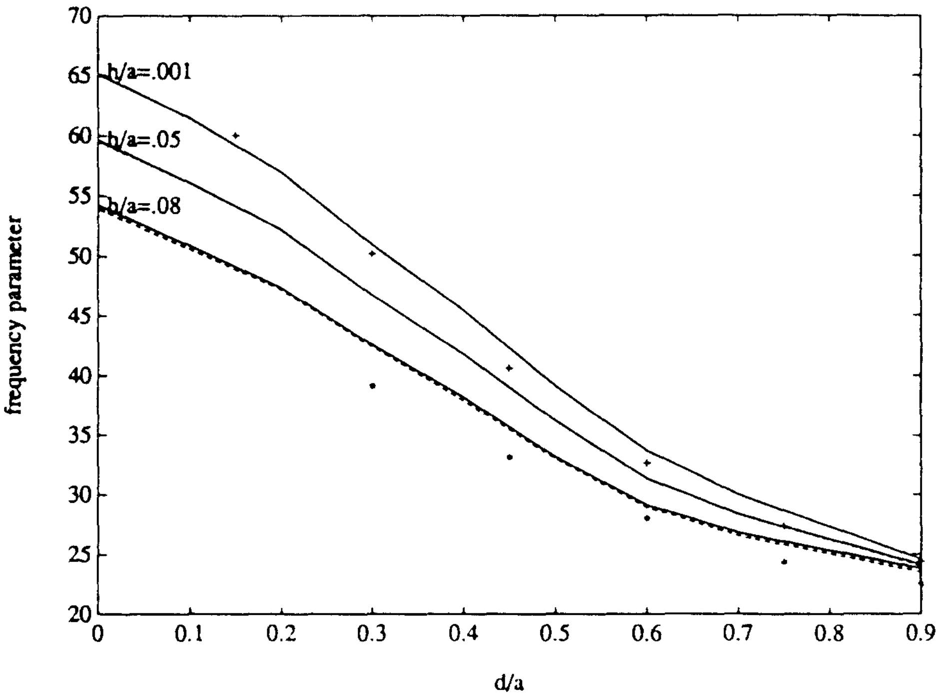

Fig. 5. Fundamental frequency parameters for a graphite-epoxy square plate having a central rectangular cutout with $d/c = 3$, "——", shear, "- - -", shear and rotary inertia, "•", $h/a = 0.1$, Reddy, "+", $h/a = 0.001$, Reddy.

G, $G/E_y = 0.5$ and $\nu_x = 0.25$, the estimated critical values of the plate thickness are presented in Table 2. The predicted natural frequencies decrease with increased plate thickness for a rectangular or square plate with the thickness less than the corresponding critical value.

The predicted natural frequencies and the reported finite element results by Reddy (1982) for graphite-epoxy orthotropic square plate with $E_x/E_y = 40$ having central rectangular and square cutouts are presented in Figs 3–5. The effect of transverse shear deformation is more pronounced for orthotropic plates with respect to the isotropic plates. The additional reduction in natural frequency caused by the effect of rotary inertia is relatively small compared with the effect of transverse shear deformation.

4. CONCLUSION

A numerical method based on the Rayleigh principle has been presented for predicting the natural frequencies of rectangular plates with rectangular cutouts allowing for both the transverse shear deformation and rotary inertia. It serves as a simple alternative to the existing analytical and finite element methods as the computation only involves the integration of simple trigonometric functions.

The present analysis is applicable for a range of thicknesses depending on the orthotropic properties and the aspect ratio of the plate. The common notion that the effect of rotary inertia is negligible with respect to the effect of transverse shear deformation is found to be valid only for orthotropic plates with a high degree of orthotropy. For isotropic plates, the effect of rotary inertia becomes more pronounced as the size of cutout increases. The additional reduction in frequency caused by the effect of rotary inertia is comparable to or even larger than the effect of transverse shear deformation for rectangular plates with large cutouts.

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